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Sophus Lie (1842-1899)

Matrix Lie Groups and Lie Algebras

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Introduction

The theory of Lie groups is a fundamental tool in theoretical physics which appears whenever a system has a continuous symmetry. We give a review of some fundamental theorems in matrix Lie groups and Lie algebras and the relationship between them.

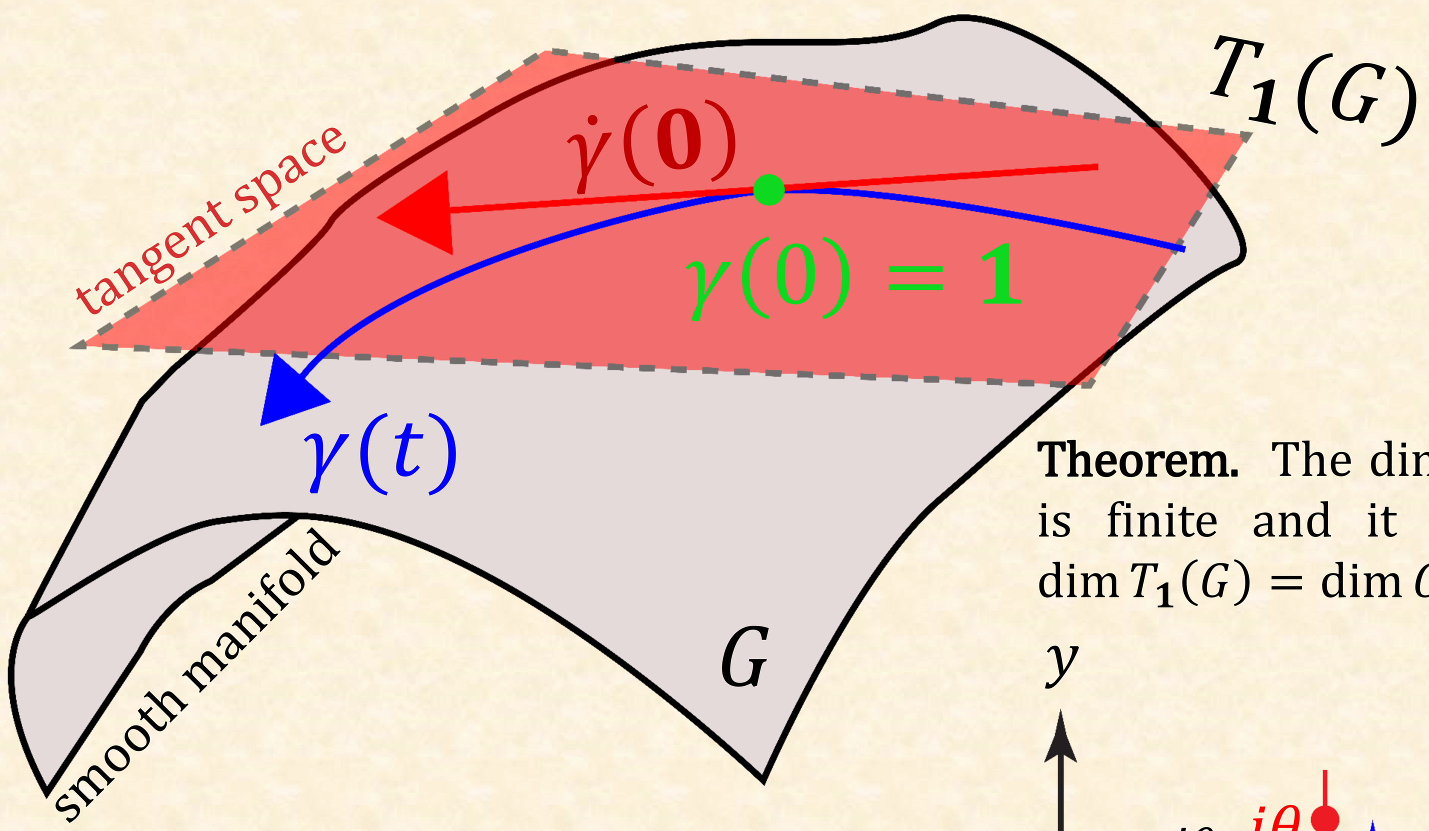
Basic concepts

Definition 1. A *matrix Lie group* is a group of matrices that is closed under nonsingular limits. That is, if $A_1, A_2, \ldots$ is a convergent sequence of matrices in G, with limit A, and if $\det(A) \neq 0$, then $A \in G$.

◆ **In a broader sense, a Lie group is a group that is also a differentiable manifold.** ◆

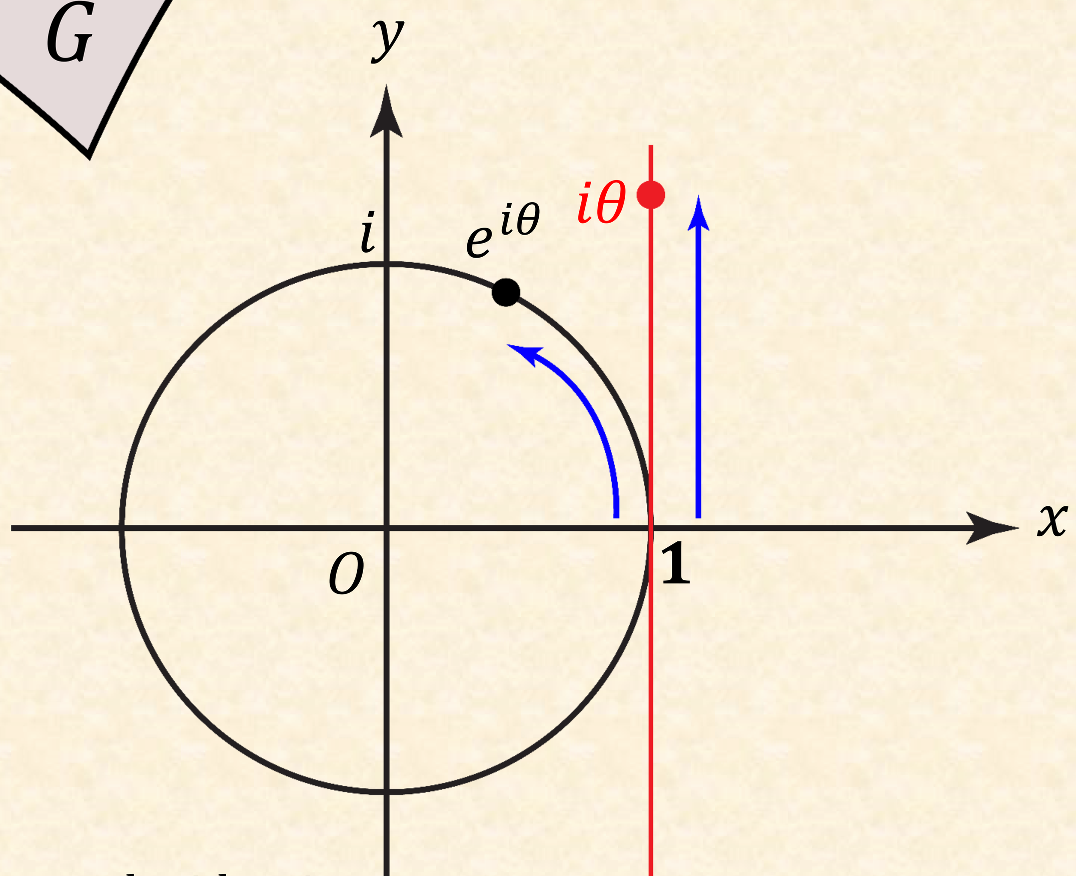
What fascinates us about Lie theory is that a Lie group G can be *almost* completely captured by the flat tangent space $T_{\mathbf{1}}(G)$ of G at the identity. $T_{\mathbf{1}}(G)$ consists of the velocity vectors of all smooth paths through $\mathbf{1}$.

Definition 2. A *matrix Lie algebra* is a vector space of matrices that is closed under the Lie bracket $[V, W] = VW - WV$.



The tangent space of G, together with its vector space structure and Lie bracket operation, is called the *Lie algebra* of G, denoted by $\mathfrak{g}$.

Theorem. The dimension of $\mathfrak{g}$ is finite and it is equal to $\dim T_{\mathbf{1}}(G) = \dim G$.



$\mathbb{S}^1$ and its tangent at the identity.

Exponential and Logarithm Maps

Exponentiation of tangent vectors. If $A'(0)$ is the tangent vector at $\mathbf{1}$ to a matrix Lie group G, then $e^{A'(0)} \in G$. That is, exp maps the tangent space $T_{\mathbf{1}}(G)$ into G.

The log of a neighbourhood of 1. For any matrix Lie group G there is a neighbourhood $N_\delta(\mathbf{1})$ mapped into $T_{\mathbf{1}}(G)$ by log.

Corollary 1. The log function gives a bijection, continuous in both direction, between $N_\delta(\mathbf{1})$ in G and $\log N_\delta(\mathbf{1})$ in $T_{\mathbf{1}}(G)$.

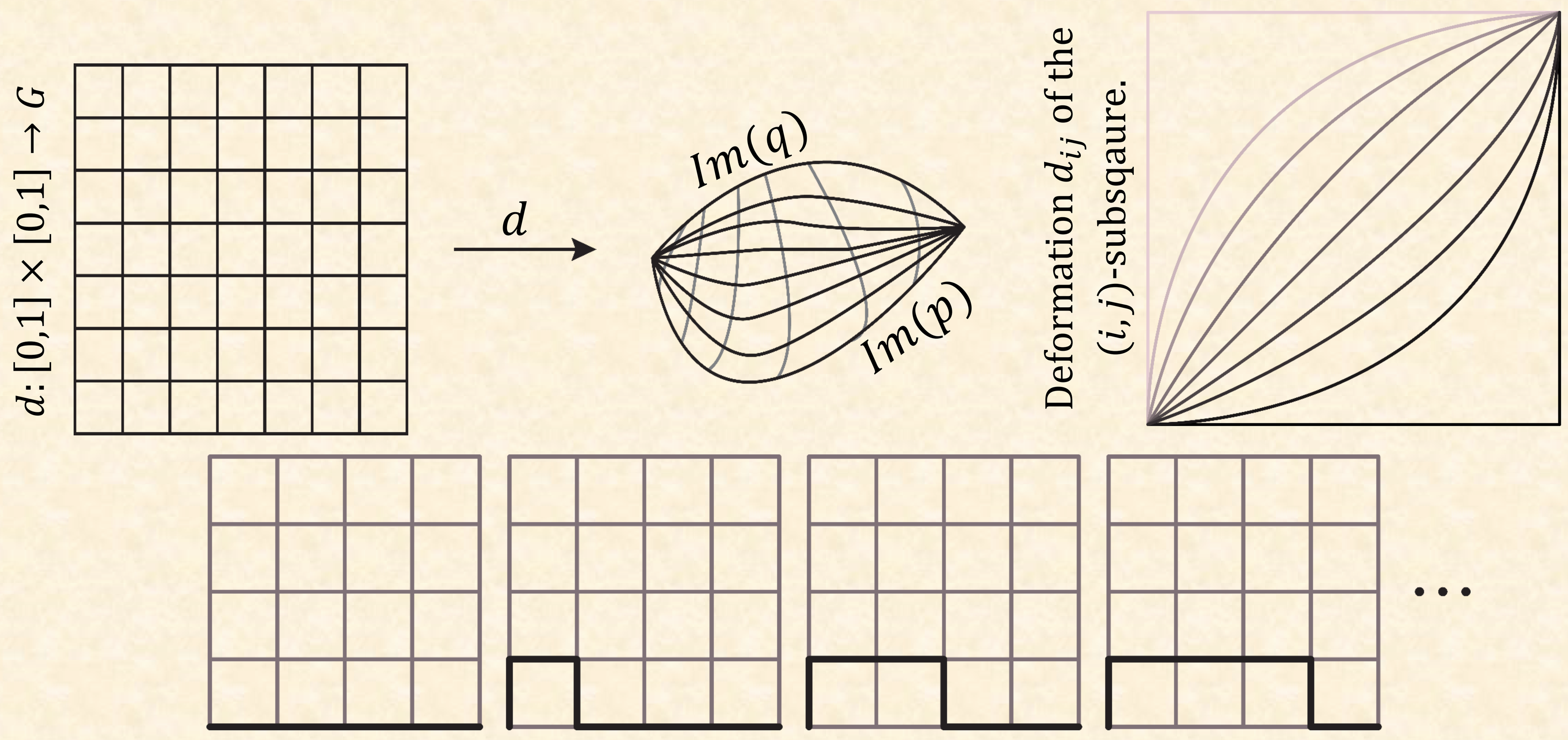
Classical groups and their Lie algebras

G	$O(n;\mathbb{R})$	$SO(n;\mathbb{R})$	$U(n)$	$SU(n)$	$Sp(n)$
$\mathfrak{g}$	$X + X^T = 0$	$X + X^T = 0$	$X + X^\dagger = 0$	$X + X^\dagger = 0$, $\operatorname{tr} X = 0$	$X + J^{-1}X^\dagger J = 0$, $X + X^\dagger = 0$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n+1)$

Campbell-Baker-Hausdorff Theorem

The Z such that $e^X e^Y = e^Z$ for possibly noncommutative X and Y in the Lie algebra of a Lie group is the sum of a series $X + Y +$ *Lie bracket terms composed from* X *and* Y. In this sense, the Lie bracket on $\mathfrak{g}$ "determines" the product operation on G.

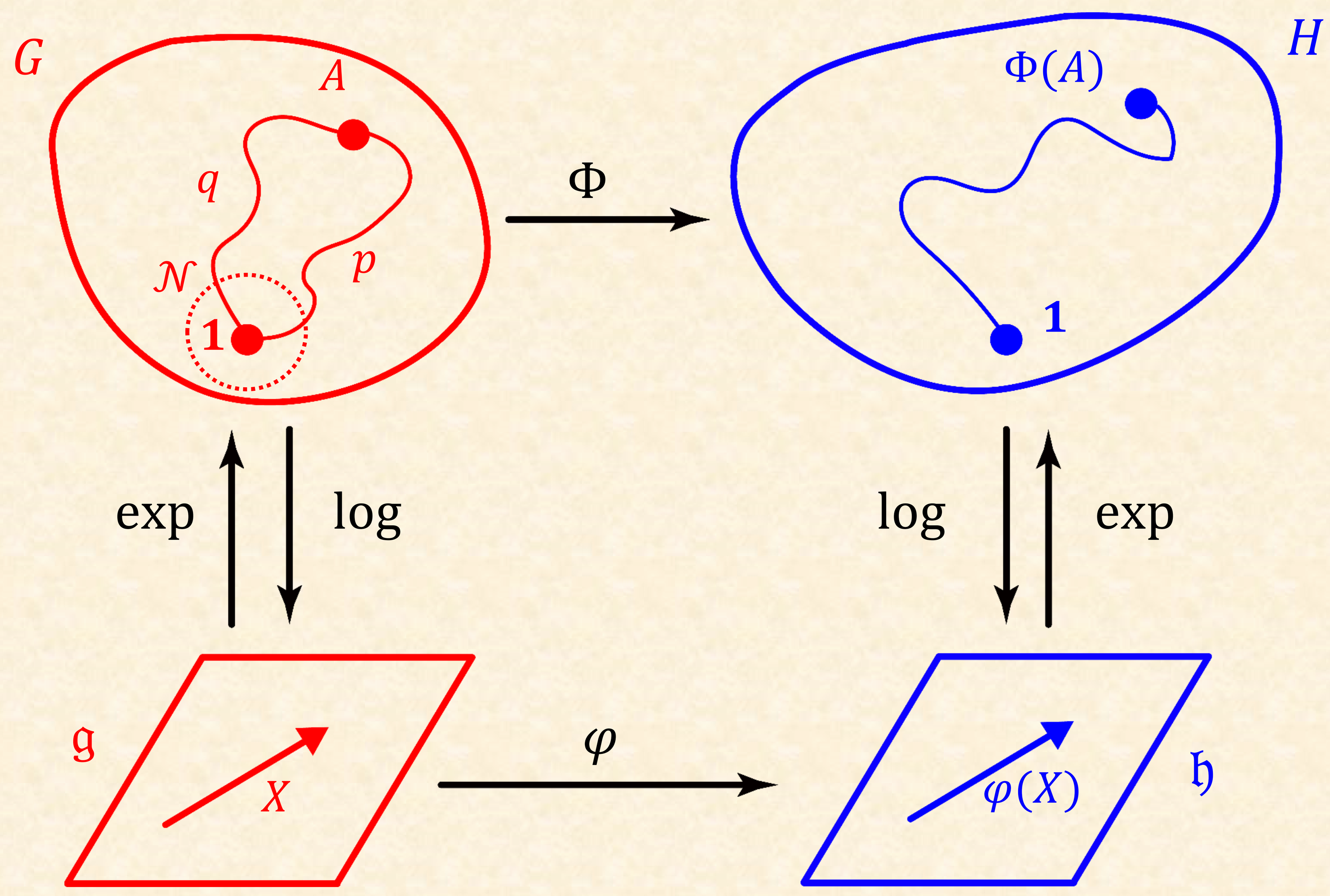
Uniform continuity of path deformation



If d is a deformation of path p to path q and d_{ij} runs through sequence of maps that deform the bottom edge of the unit square to the top, then the sequence of composite maps $d \circ d_{ij}$ deforms p to q, and each $d \circ d_{ij}$ agrees with d outside a neighbourhood of the image of the (i,j)-subsquare, and hence outside an ε-ball. In this sense, if a path p can be deformed to a path q, then p can be deformed to q in a finite sequence of "small" steps.

Lifting a Lie Algebra Homomorphism

If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of simply connected Lie groups G and H, respectively, then each Lie algebra homomorphism $\varphi: \mathfrak{g} \to \mathfrak{h}$ is induced by a Lie group homomorphism $\Phi: G \to H$.



Corollary 2. If G and H are simply connected Lie groups with isomorphic Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, then G is isomorphic to H.